The Structure of Multibody Dynamics Equations

W. Jerkovsky*
The Aerospace Corporation, El Segundo, Calif.

Several alternative formulations for the dynamics of multibody systems are described. These alternatives include momentum and velocity formulations with decoupling or coupling of constraints. The presentation of equations is facilitated by the introduction of a path matrix and a reference matrix that describe the topology of the n-body configurations. The final equations of motion are obtained from the equations of motion for a single body via the transformation operator formalism. The equations of motion presented herein are compared with 10 n-body dynamics formulations in the spacecraft dynamics literature.

I. Introduction

THE purpose of this paper is to present an overview of several n-body dynamics formulations in the spacecraft dynamics literature. Even though the emphasis is different, the overview in this paper is somewhat in the spirit of Likins[1-3] and Meirovitch,[4] and the background references for this paper are essentially the same as for these earlier overviews. This paper differs from previous papers in the spacecraft dynamics literature in that both "momentum formulations" and "velocity formulations" are discussed in a single language: the language of the transformation operator formalism.[5]

The paper starts out with a description of multibody tree configurations. A path matrix π and a reference matrix ρ are defined in the spirit of Roberson and Wittenburg.[6] The next step is the introduction of the "primitive" or "free body" equations of motion in terms of a single equation. This equation then is transformed linearly via a transformation operator A, and the result is a new "transformed" equation of motion. This method of transforming "old" differential equations to "new" differential equations is based on Kron's method of subspaces[7] and is similar to the matrix method of structural analysis.[8] In the old differential equations the velocities are inertial velocities, whereas in the new differential equations the velocities are relative velocities. The transformation to relative velocities is made so that relative velocity constraints can be treated more readily; this transformation is made in the spirit of classical mechanics,[9] where generalized coordinates are introduced so that the constraints become trivial.

As an alternative to transforming to relative velocities, the equations of motion can be kept in terms of inertial velocities, and the relative velocity constraints then can be incorporated via Lagrange multipliers. This alternate approach is particularly attractive in cases where it is not a simple matter to express the inertial velocities of a multibody system in terms of an independent set of relative velocities; such a case occurs when the multibody configuration is not a tree, i.e., when there are closed loops.

The equations presented in this paper assume that the multibody configuration consists of n rigid bodies. The same procedure can be used if some or all of the bodies are flexible; i.e., the structure of the multibody dynamics equations is the same whether the bodies are flexible or rigid. In fact, a large number of rigid bodies can be used to model a flexible body, and the structure of the equations does not depend on whether n is large. If it is desired to treat all n bodies as flexible, then there are three modifications that are required: 1) the "primitive" or "free body" equation of motion must include equations of motion for the deformation degrees of freedom[10,11]; 2) the "primitive" and "transformed" velocities must include the time derivatives of the deformation coordinates[12]; and 3) the transformation operator A must express the inertial velocities in terms of relative velocities plus deformation coordinates time derivatives. An introduction to the treatment of flexible bodies via the transformation operator formalism is given by Russell.[13]

II. Description of Multibody Tree Configurations

Given an n-body configuration, label the bodies from 1 to n, assigning the label 1 to the "main" or "central" body. We obtain a "graph" of the configuration by putting each body in correspondence with a vertex (or node) of a graph and connecting any two vertices of this graph with a branch if the corresponding bodies have any degrees of relative motion between them. If the resulting graph is a tree (i.e., if there are no closed loops), then the n-body system is said to have a tree configuration. If the graph is not a tree, then a tree still can be associated with the graph by cutting as many branches as there are closed loops. Any branch in any closed loop may be cut, and different choices will lead to different trees of the graph.

Thus, to any n-body configuration there corresponds a tree with Body 1 at the center of the tree. For the moment, we do not concern ourselves with how many degrees of freedom there are between adjacent vertices (i.e., between adjacent bodies), or if the actual n-body configuration has closed loops or not.

Label the bodies (or vertices of the tree) such that all the bodies between Body 1 and Body j have an index i between 1 and j. Also, let $\hat{j}$ be the set of integers which includes 1 and j and also includes the labels of all bodies between Body 1 and Body j. Let $\underline{j}$ be the label of the body next to Body j on the path from Body j to Body 1; similarly, let $\underline{\underline{j}}$ be the label of the body next to Body $\underline{j}$; etc. Then, the set $\hat{j}$ consists of the labels $\hat{j} = \{j, \underline{j}, \underline{\underline{j}}, \ldots, 1\}$. Evidently, $\hat{j}$ is the set of indices of all bodies that are "inward" from Body j, including Body j. Let $\underline{1} = 0$ so that $\underline{j}$ is defined for $j = 1$ to n.

Next, let $\bar{k}$ be the set of all j such that k is contained in $\hat{j}$ (i.e., such that $k \in \hat{j}$). Evidently, $\bar{k}$ is the set of indices of all bodies that are "outward" from Body k, including Body k. Note that the set $\bar{1}$ includes the integers from 1 to n ($\bar{1} = \{1, 2, \ldots, n\}$) because all bodies are outward from Body 1.

Now introduce the "path matrix" π as follows. Letting π_{ij} denote the element of π in the ith row and jth column, define

$$\pi_{ij} = \begin{cases} 1 & \text{if Body } j \text{ is between Body 1 and Body } i \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

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*Member of Technical Staff, Guidance and Control Division.

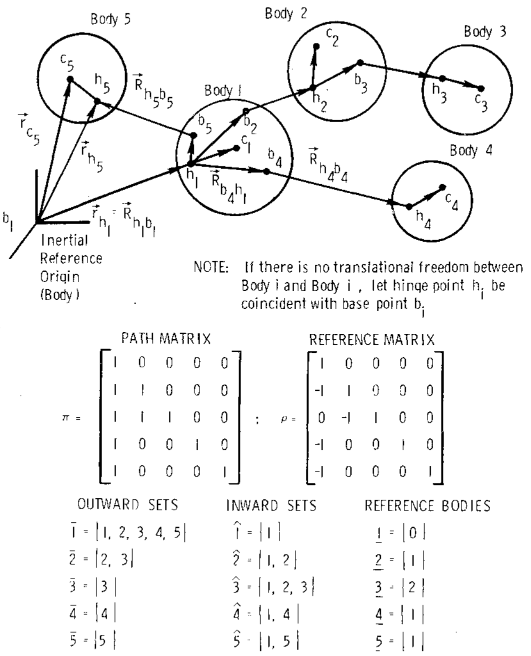

Fig. 1 Example of a five-body configuration.

Note that the rows of π are determined by the "inward" set $\hat{i}$ for $i=1$ to n, and the columns of π are determined by the "outward" sets $\bar{j}$ (see Fig. 1).

If we think of Body $\underset{-}{j}$ as the body to which Body j is "referenced," then we can define the "reference matrix" ρ as follows:

$$\rho_{ij} = \begin{cases} 1 & \text{if } i=j \\ -1 & \text{if Body } i \text{ is referenced to Body } \underset{-}{j} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Evidently, the -1 terms of ρ can be filled in by looking at the labels $\underset{-}{2}, \underset{-}{3}, \ldots, \underset{-}{n}$. Note that the labels $\underset{-}{2}, \underset{-}{3}, \ldots, \underset{-}{n}$ define the tree configuration completely, because these labels determine the tree and the sets $\hat{j}$ and $\bar{k}$ for $j, k=1, 2, \ldots, n$. Also note that column j of ρ shows which bodies are referenced to Body j.

The matrices π and ρ are lower triangular, and ρ is sparser than π. However, π and ρ contain the same information; in fact, π and ρ are inverses of each other:

$$\rho\pi = 1_n = \pi\rho \tag{3}$$

where 1_n is the $n \times n$ identity matrix. We shall see later that, after the matrices π or ρ are introduced for a particular tree configuration, the equations become independent of the particular configuration under consideration.

Define the set $\overline{ij}$ to be the intersection of the sets $\bar{i}$ and $\bar{j}$:

$$\overline{ij} = \bar{i} \cap \bar{j} = \overline{ji} \tag{4}$$

Now note that all of the sets $\bar{k}$, for $k=1$ to n, are "nested" in the sense that, if the sets $\bar{i}$ and $\bar{j}$ have any elements in common, then either $\bar{i}$ is contained in $\bar{j}$, or $\bar{j}$ is contained in $\bar{i}$. Hence, $\overline{ij}$ is either $\bar{i}$ or $\bar{j}$ or the empty set $\emptyset$. Symbolically,

$$\overline{ij} = \begin{cases} \bar{i} & \text{if } \bar{i} \subset \bar{j} \\ \bar{j} & \text{if } \bar{j} \subset \bar{i} \\ \emptyset & \text{otherwise} \end{cases} \tag{5}$$

Note that $\overline{ii} = \bar{i}$.

Now introduce the points c_i, h_i, and b_i as follows, for $i=1$ to n. Let c_i be the center of mass of Body i. If Body i is a rigid body, then the point c_i is a fixed material point of Body i; if Body i is deformable, then c_i "floats" in the body. Whether Body i is rigid or deformable, let h_i be a fixed material point that is the "hinge" point for body i. Let b_i be a fixed material point in Body $\underset{-}{i}$ to which Body i is referenced; the point b_i is the "base" point for Body i. By convention, b_1 is the inertial reference origin, i.e., a point fixed in "Body 0" (see Fig. 1).

III. Primitive Equations of Motion

By "primitive" equations and variables, we mean equations and variables that refer to each body as a separate and distinct body without regard to how it fits into the multibody configuration. Let $\boldsymbol{P}^i$ be the linear momentum of Body i, and let $\boldsymbol{F}^i$ be the force on Body i. Let $\boldsymbol{H}^i_{c_i}$ be the angular momentum of Body i about c_i, and let $\boldsymbol{L}^i_{c_i}$ be the torque on Body i about c_i. Note that we are using the body index i as a superscript and the point c_i as a subscript. Let $\boldsymbol{v}_{c_i}$ be the linear velocity of the point c_i; thus, if $\boldsymbol{r}_{c_i}$ is the position vector to c_i from the inertial reference origin, then $\boldsymbol{v}_{c_i} = \dot{\boldsymbol{r}}_{c_i}$, where the dot over a vector is used to denote the time derivative in the inertial reference frame. Let $\boldsymbol{\omega}^i$ be the angular velocity of a frame fixed in Body i. For simplicity, we now assume that all bodies are rigid; the extension to the case where some or all bodies are deformable is beyond the scope of this paper. Let M^i be the mass of Body i, and let $\vec{I}^i_{c_i}$ be the inertia (dyadic) of Body i about c_i; let $W^i = (M^i)^{-1}$ and $\vec{J}^i_{c_i} = (\vec{I}^i_{c_i})^{-1}$. Now define G, K, σ, and Y to be column matrices of $2n$ Gibbsian vector as follows[5]:

$$G = \begin{bmatrix} \boldsymbol{H}^1_{c_1} \\ \boldsymbol{H}^2_{c_2} \\ \vdots \\ \boldsymbol{H}^n_{c_n} \\ \boldsymbol{P}^1 \\ \boldsymbol{P}^2 \\ \vdots \\ \boldsymbol{P}^n \end{bmatrix}, \quad K = \begin{bmatrix} \boldsymbol{L}^1_{c_1} \\ \boldsymbol{L}^2_{c_2} \\ \vdots \\ \boldsymbol{L}^n_{c_n} \\ \boldsymbol{F}^1 \\ \boldsymbol{F}^2 \\ \vdots \\ \boldsymbol{F}^n \end{bmatrix}, \quad \sigma = \begin{bmatrix} \boldsymbol{\omega}^1 \\ \boldsymbol{\omega}^2 \\ \vdots \\ \boldsymbol{\omega}^n \\ \boldsymbol{v}_{c_1} \\ \boldsymbol{v}_{c_2} \\ \vdots \\ \boldsymbol{v}_{c_n} \end{bmatrix}, \quad Y = \begin{bmatrix} \boldsymbol{E}^1_{c_1} \\ \boldsymbol{E}^2_{c_2} \\ \vdots \\ \boldsymbol{E}^n_{c_n} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \end{bmatrix} \tag{6}$$

where $\boldsymbol{0}$ is the zero vector, and $\boldsymbol{E}^i_{c_i}$ is the "Euler coupling force" on Body i: $\boldsymbol{E}^i_{c_i} = \boldsymbol{\omega}^i \times \boldsymbol{H}^i_{c_i}$. Also, define μ and ν as diagonal matrices of positive definite symmetric dyadics as follows:

$$\mu = \mathrm{diag}[\ \vec{I}^1_{c_1},\ \vec{I}^2_{c_2},\ \ldots,\ \vec{I}^n_{c_n},\ \vec{M}^1,\ \vec{M}^2,\ \ldots,\ \vec{M}^n\ ] \tag{7a}$$

$$\nu = \mathrm{diag}[\ \vec{J}^1_{c_1},\ \vec{J}^2_{c_2},\ \ldots,\ \vec{J}^n_{c_n},\ \vec{W}^1,\ \vec{W}^2,\ \ldots,\ \vec{W}^n\ ] \tag{7b}$$

where $\vec{M}^i = M^i\ \vec{1}$, $\vec{W}^i = W^i\ \vec{1}$ and $\vec{1}$ is the identity dyadic. The off-diagonal elements of μ and ν are the zero dyadic $\vec{O}$. The primitive momentum formulation equations for the n-body system now are given by

$$\dot{G} + X = K, \qquad G = \mu \cdot \sigma \quad \text{or} \quad \sigma = \nu \cdot G \tag{8}$$

where X is a column matrix of zero vectors which is included here only for pedagogical reasons. G, K, and σ will be called the primitive system momentum, force, and velocity, respectively. μ and ν will be called the primitive system mass and inverse mass, respectively. The velocity formulation equation is

$$\mu \cdot \dot{\sigma} + Y = K \tag{9}$$

The kinetic energy of the system of n rigid bodies is given by

$$T = \tfrac{1}{2} G^t \cdot \sigma = \tfrac{1}{2} \sigma^t \cdot \mu \cdot \sigma = \tfrac{1}{2} G^t \cdot \nu \cdot G \qquad (10)$$

where G^t and σ^t are the transposes of G and σ; thus,

$$G^t = [\boldsymbol{H}^1_{c_1} \quad \boldsymbol{H}^2_{c_2} \ldots \boldsymbol{H}^n_{c_n} \quad \boldsymbol{P}^1 \quad \boldsymbol{P}^2 \ldots \boldsymbol{P}^n]$$

and similarly for σ^t. The time derivative of the kinetic energy is given by

$$\dot{T} = K^t \cdot \sigma \qquad (11)$$

In the next section, we express the primitive system velocity σ in terms of a new (or transformed) system velocity $\bar{\sigma}$. This then automatically defines new system variables so that Eqs. (8-11) maintain their form.

IV. Transformed Equations of Motion

We ultimately shall be interested in the case where there are less than six free degrees of freedom between some or all of the adjacent bodies of the tree configuration. Therefore, we now shall transform to relative velocities, which then can be prescribed if the corresponding degrees of freedom are constrained.

A. Velocity Transformation

Now express the inertial velocities $\boldsymbol{\omega}^i$ and $\boldsymbol{v}_{c_i}$ in terms of the relative velocities $\boldsymbol{\Omega}^i$ and $\boldsymbol{U}^i$. Define $\boldsymbol{\Omega}^i$ as the angular velocity of Body i with respect to Body $\underline{i}$ to which it is referenced:

$$\boldsymbol{\Omega}^i = \boldsymbol{\omega}^i - \boldsymbol{\omega}^{\underline{i}} \qquad (12)$$

"Body 0" is the inertial reference frame, and hence $\boldsymbol{\omega}^0 = \boldsymbol{0}$. Therefore, $\boldsymbol{\Omega}^1 = \boldsymbol{\omega}^1$.

Let $\boldsymbol{R}_{ab} = \boldsymbol{r}_a - \boldsymbol{r}_b$ be the position vector to point a from point b. Also, for $k=1$ to n, let $\overset{k}{\dot{V}}$ denote the time derivative of the vector $\boldsymbol{V}$ with respect to Body k; then, $\overset{0}{\dot{V}} = \dot{V}$. Now define $\boldsymbol{U}^i$ as the time derivative with respect to Body $\underline{i}$ of the position vector to the hinge point h_i in Body i from the base point b_i in Body $\underline{i}$:

$$\boldsymbol{U}^i = \overset{\underline{i}}{\dot{\boldsymbol{R}}}_{h_i b_i} \qquad (13)$$

Note that $\boldsymbol{U}^1 = \dot{\boldsymbol{R}}_{h_1 b_1} = \boldsymbol{v}_{h_1}$, since $\boldsymbol{r}_{b_1} = \boldsymbol{0}$.

The primitive inertial velocities $\boldsymbol{\omega}^i$ and $\boldsymbol{v}_{c_i}$ can be expressed in terms of the transformed relative velocities as follows[14]:

$$\boldsymbol{\omega}^i = \sum_{j=1}^{n} \pi_{ij} \boldsymbol{\Omega}^j = \sum_{j \in \hat{i}} \boldsymbol{\Omega}^j \qquad (14a)$$

$$\boldsymbol{v}_{c_i} = \sum_{j=1}^{n} \pi_{ij} (\tilde{\boldsymbol{R}}^t_{c_i h_j} \cdot \boldsymbol{\Omega}^j + \boldsymbol{U}^j) = \sum_{j \in \hat{i}} (\tilde{\boldsymbol{R}}^t_{c_i h_j} \cdot \boldsymbol{\Omega}^j + \boldsymbol{U}^j) \qquad (14b)$$

where we make use of the notation that, for any two vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, the dyadic of $\boldsymbol{\alpha}$ is denoted by $\tilde{\alpha}$ and is defined by $\tilde{\alpha} \cdot \boldsymbol{\beta} = \boldsymbol{\alpha} \times \boldsymbol{\beta}$, and $\tilde{\alpha}^t$ is the dyadic transpose of $\tilde{\alpha}$ and is defined by $\tilde{\alpha} \cdot \boldsymbol{\beta} = \boldsymbol{\beta} \cdot \tilde{\alpha}^t$; note that $\tilde{\alpha}^t = -\tilde{\alpha}$, i.e., $\tilde{\alpha}$ is skew-symmetric. The inverses of Eqs. (14) are

$$\boldsymbol{\Omega}^i = \sum_{j=1}^{n} \rho_{ij} \boldsymbol{\omega}^j = \boldsymbol{\omega}^i - \boldsymbol{\omega}^{\underline{i}} \qquad (15a)$$

$$\boldsymbol{U}^i = \sum_{j=1}^{n} \rho_{ij} (\tilde{\boldsymbol{R}}_{c_j h_i} \cdot \boldsymbol{\omega}^j + \boldsymbol{v}_{c_j})$$

$$= \tilde{\boldsymbol{R}}_{c_i h_i} \cdot \boldsymbol{\omega}^i + \boldsymbol{v}_{c_i} - \tilde{\boldsymbol{R}}_{c_{\underline{i}} h_i} \cdot \boldsymbol{\omega}^{\underline{i}} - \boldsymbol{v}_{c_{\underline{i}}} \qquad (15b)$$

Note that Eq. (15a) is the same as Eq. (12).

We now define $\bar{\sigma}$ to be a column matrix of vectors (and, therefore, $\bar{\sigma}^t$ is a row matrix of vectors) as follows:

$$\bar{\sigma}^t = [\boldsymbol{\Omega}^1 \quad \boldsymbol{\Omega}^2 \cdots \boldsymbol{\Omega}^n \quad \boldsymbol{U}^1 \quad \boldsymbol{U}^2 \cdots \boldsymbol{U}^n] \qquad (16)$$

Equations (14) and (15) now can be written in the form

$$\sigma = A \cdot \bar{\sigma}, \qquad \bar{\sigma} = B \cdot \sigma \quad \text{where} \quad B = A^{-1} \qquad (17)$$

A is the transformation operator that expresses the primitive system velocity σ in terms of the transformed system velocity $\bar{\sigma}$, and B is its inverse. A and B are matrices of dyadics; their elements can be obtained by inspection of Eqs. (14) and (15). Note that $A \cdot B = B \cdot A$ is an identity with $2n$ $\overleftrightarrow{1}$'s on the diagonal (see Refs. 5 and 14 for examples of A and B matrices).

B. Induced Transformations

In order for Eqs. (10) and (11) to maintain their form, we define the transformed momentum, force, and mass as follows:

$$\bar{G} = A^t \cdot G, \qquad \bar{K} = A^t \cdot K, \qquad \bar{\mu} = A^t \cdot \mu \cdot A \qquad (18)$$

Carrying out the operation $A^t \cdot G$, we find $\bar{G}$ as follows:

$$\bar{G}^t = [\boldsymbol{H}^{\bar{1}}_{h_1} \quad \boldsymbol{H}^{\bar{2}}_{h_2} \cdots \boldsymbol{H}^{\bar{n}}_{h_n} \quad \boldsymbol{P}^{\bar{1}} \quad \boldsymbol{P}^{\bar{2}} \cdots \boldsymbol{P}^{\bar{n}}] \qquad (19)$$

where

$$\boldsymbol{H}^{\bar{j}}_{h_j} = \sum_{i=1}^{n} \pi_{ij} (\boldsymbol{H}^i_{c_i} + \tilde{\boldsymbol{R}}_{c_i h_j} \cdot \boldsymbol{P}^i) = \sum_{i \in \bar{j}} (\boldsymbol{H}^i_{c_i} + \tilde{\boldsymbol{R}}_{c_i h_j} \cdot \boldsymbol{P}^i) \qquad (20a)$$

$$\boldsymbol{P}^{\bar{j}} = \sum_{i=1}^{n} \pi_{ij} \boldsymbol{P}^i = \sum_{i \in \bar{j}} \boldsymbol{P}^i \qquad (20b)$$

Evidently, $\boldsymbol{P}^{\bar{j}}$ is the linear momentum of the set of bodies whose labels are in the set $\bar{j}$; i.e., the set of bodies outward from Body j, including Body j. If we refer to this set of bodies as "System $\bar{j}$," then $\boldsymbol{P}^{\bar{j}}$ is the linear momentum of System $\bar{j}$. Similarly, $\boldsymbol{H}^{\bar{j}}_{h_j}$ is the angular momentum of System $\bar{j}$ about the hinge point h_j in Body j. Carrying out the operation $A^t \cdot K$ yields the same type of equations, with $\boldsymbol{L}$'s and $\boldsymbol{F}$'s replacing $\boldsymbol{H}$'s and $\boldsymbol{P}$'s, respectively.

The definitions of $\bar{G}$ and $\bar{\mu}$ given in Eq. (18) now yield $\bar{G} = \bar{\mu} \cdot \bar{\sigma}$ in the form

$$\begin{bmatrix} \boldsymbol{H}^{\bar{1}}_{h_1} \\ \vdots \\ \boldsymbol{H}^{\bar{n}}_{h_n} \\ \boldsymbol{P}^{\bar{1}} \\ \vdots \\ \boldsymbol{P}^{\bar{n}} \end{bmatrix} = \begin{bmatrix} \overleftrightarrow{\boldsymbol{I}}^{\bar{1}\bar{1}}_{h_1 h_1} & \cdots & \overleftrightarrow{\boldsymbol{I}}^{\bar{1}\bar{n}}_{h_1 h_n} & \overleftrightarrow{\boldsymbol{S}}_{c_{\bar{1}\bar{1}} h_1} & \cdots & \overleftrightarrow{\boldsymbol{S}}_{c_{\bar{1}\bar{n}} h_1} \\ \vdots & & \vdots & \vdots & & \vdots \\ \overleftrightarrow{\boldsymbol{I}}^{\bar{n}\bar{1}}_{h_n h_1} & \cdots & \overleftrightarrow{\boldsymbol{I}}^{\bar{n}\bar{n}}_{h_n h_n} & \overleftrightarrow{\boldsymbol{S}}_{c_{\bar{n}\bar{1}} h_n} & \cdots & \overleftrightarrow{\boldsymbol{S}}_{c_{\bar{n}\bar{n}} h_n} \\ \overleftrightarrow{\boldsymbol{S}}^t_{c_{\bar{1}\bar{1}} h_1} & \cdots & \overleftrightarrow{\boldsymbol{S}}^t_{c_{\bar{n}\bar{1}} h_n} & \overleftrightarrow{\boldsymbol{M}}^{\bar{1}\bar{1}} & & \overleftrightarrow{\boldsymbol{M}}^{\bar{1}\bar{n}} \\ \vdots & & \vdots & \vdots & & \vdots \\ \overleftrightarrow{\boldsymbol{S}}^t_{c_{\bar{1}\bar{n}} h_1} & \cdots & \overleftrightarrow{\boldsymbol{S}}^t_{c_{\bar{n}\bar{n}} h_n} & \overleftrightarrow{\boldsymbol{M}}^{\bar{n}\bar{1}} & \cdots & \overleftrightarrow{\boldsymbol{M}}^{\bar{n}\bar{n}} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{\Omega}^1 \\ \vdots \\ \boldsymbol{\Omega}^n \\ \boldsymbol{U}^1 \\ \vdots \\ \boldsymbol{U}^n \end{bmatrix} \qquad (21)$$

where

$$\overleftrightarrow{\boldsymbol{I}}^{\bar{i}\bar{j}}_{h_i h_j} = \sum_{k=1}^{n} \pi_{ki} \pi_{kj} (\overleftrightarrow{\boldsymbol{I}}^k_{c_k} + M^k \tilde{\boldsymbol{R}}_{c_k h_i} \cdot \tilde{\boldsymbol{R}}^t_{c_k h_j})$$

$$= \sum_{k \in \bar{i}\bar{j}} (\overleftrightarrow{\boldsymbol{I}}^k_{c_k} + M^k \tilde{\boldsymbol{R}}_{c_k h_i} \cdot \tilde{\boldsymbol{R}}^t_{c_k h_j}) \qquad (22a)$$

$$\overleftrightarrow{\boldsymbol{S}}_{c_{\bar{i}\bar{j}} h_i} = M^{\bar{i}\bar{j}} \tilde{\boldsymbol{R}}_{c_{\bar{i}\bar{j}} h_i} = \sum_{k=1}^{n} \pi_{ki} \pi_{kj} M^k \tilde{\boldsymbol{R}}_{c_k h_i} = \sum_{k \in \bar{i}\bar{j}} M^k \tilde{\boldsymbol{R}}_{c_k h_i} \qquad (22b)$$

$$\overleftrightarrow{\boldsymbol{M}}^{\bar{i}\bar{j}} = M^{\bar{i}\bar{j}} \overleftrightarrow{\boldsymbol{1}} = \sum_{k=1}^{n} \pi_{ki} \pi_{kj} M^k \, \overleftrightarrow{\boldsymbol{1}} = \sum_{k \in \bar{i}\bar{j}} M^k \, \overleftrightarrow{\boldsymbol{1}} \qquad (22c)$$

Recall that the set $\overline{ij}$ is the intersection of the sets $\bar{i}$ and $\bar{j}$. If $\overline{ij}=\emptyset$, then the sum over $k\in\overline{ij}$ yields the zero dyadic $\vec{\vec{O}}$. If $i=j$, then $\vec{\vec{I}}^{\bar{j}}_{h_j h_j}=\vec{\vec{I}}^{\bar{j}}_{h_j}$, which is the inertia (dyadic) of System $\bar{j}$ about h_j; also, $M^{\bar{j}\bar{j}}=M^{\bar{j}}$, which is the mass of System $\bar{j}$. Note that $R_{c_{\overline{ij}}h_i}$ is the position vector to the point $c_{\overline{ij}}$, the center of mass of System $\overline{ij}$, from the point h_i (for an example of $\bar{\mu}$, see Refs. 5 and 14).

Equations (18) have the inverse relationships

$$G=B^t\bullet\bar{G},\qquad K=B^t\bullet\bar{K},\qquad \bar{\nu}=B\bullet\nu\bullet B^t \tag{23}$$

Carrying out the operation $B^t\bullet\bar{G}$ yields the inverse of Eqs. (20):

$$H^j_{c_j}=\sum_{i=1}^{n}\rho_{ij}(H^{\bar{i}}_{h_i}+\vec{R}_{h_i c_j}\bullet P^{\bar{i}}),\quad P^j=\sum_{i=1}^{n}\rho_{ij}P^{\bar{i}} \tag{24}$$

Similar results are obtained from $B^t\bullet\bar{K}$, with L's and F's replacing H's and P's, respectively. The relationship $\bar{\sigma}=\bar{\nu}\bullet G$ takes the form

$$\begin{bmatrix}\Omega^1\\ \vdots\\ \Omega^n\\ U^1\\ \vdots\\ U^n\end{bmatrix}=\begin{bmatrix}\vec{\vec{J}}^{11} & \cdots & \vec{\vec{J}}^{1n} & \vec{\vec{Z}}^{11} & \cdots & \vec{\vec{Z}}^{1n}\\ \vdots & & \vdots & \vdots & & \vdots\\ \vec{\vec{J}}^{n1} & \cdots & \vec{\vec{J}}^{nn} & \vec{\vec{Z}}^{n1} & \cdots & \vec{\vec{Z}}^{nn}\\ \vec{\vec{Z}}^{11t} & \cdots & \vec{\vec{Z}}^{n1t} & \vec{\vec{W}}^{11} & \cdots & \vec{\vec{W}}^{1n}\\ \vdots & & \vdots & & & \vdots\\ \vec{\vec{Z}}^{1nt} & \cdots & \vec{\vec{Z}}^{nnt} & \vec{\vec{W}}^{n1} & \cdots & \vec{\vec{W}}^{nn}\end{bmatrix}\bullet\begin{bmatrix}H^{\bar{1}}_{h_1}\\ \vdots\\ H^{\bar{n}}_{h_n}\\ P^{\bar{1}}\\ \vdots\\ P^{\bar{n}}\end{bmatrix} \tag{25}$$

where

$$\vec{\vec{J}}^{ik}=\sum_{j=1}^{n}\rho_{ij}\rho_{kj}\vec{\vec{J}}^j_{c_j} \tag{26a}$$

$$\vec{\vec{Z}}^{ik}=\sum_{j=1}^{n}\rho_{ij}\rho_{kj}\vec{\vec{J}}^j_{c_j}\bullet\vec{R}^t_{c_j h_k} \tag{26b}$$

$$\vec{\vec{W}}^{ik}=\sum_{j=1}^{n}\rho_{ij}\rho_{kj}(\vec{R}_{c_j h_i}\bullet\vec{\vec{J}}^j_{c_j}\bullet\vec{R}^t_{c_j h_k}+W^j\vec{\vec{1}}) \tag{26c}$$

Since the reference matrix ρ is generally sparser than the path matrix π, there are fewer terms involved in generating the elements of $\bar{\nu}$ than in generating the elements of $\bar{\mu}$.

C. Transformed Equations of Motion

The transformed momentum formulation equation of motion is

$$\dot{\bar{G}}+\bar{X}=\bar{K}\qquad\text{where}\qquad \bar{X}=A^t\bullet X-\dot{A}^t\bullet G \tag{27}$$

and the transformed velocity formulation equation of motion is

$$\bar{\mu}\bullet\dot{\bar{\sigma}}+\bar{Y}=\bar{K}\quad\text{where}\quad \bar{Y}=A^t\bullet(Y+\mu\bullet\dot{A}\bullet\bar{\sigma}) \tag{28}$$

Performing the indicated operations for $\bar{X}$ and $\bar{Y}$ yields

$$\bar{X}^t=[\tilde{v}_{h_1}\bullet P^{\bar{1}}\ldots\tilde{v}_{h_n}\bullet P^{\bar{n}}\ \mathbf{0}\ldots\mathbf{0}] \tag{29a}$$

$$\bar{Y}^t=[\bar{E}^{\bar{1}}_{h_1}\ldots\bar{E}^{\bar{n}}_{h_n}\ \bar{C}^{\bar{1}}\ldots\bar{C}^{\bar{n}}] \tag{29b}$$

where

$$\bar{E}^{\bar{j}}_{h_j}=\sum_{i\in\bar{j}}(E^i_{c_i}+\tilde{R}_{c_i h_j}\bullet\bar{C}^i) \tag{30}$$

$$\bar{C}^i\ =M^i\bar{a}_{c_i}=M^i\sum_{k\in\hat{i}}\dot{\tilde{R}}^t_{c_i h_k}\bullet\Omega^k \tag{31}$$

$$\bar{C}^{\bar{j}}=\sum_{i\in\bar{j}}\bar{C}^i=\sum_{k=1}^{n}\dot{\tilde{S}}^t_{c_{\overline{kj}}h_k}\bullet\Omega^k \tag{32}$$

From Eq. (14b) for v_{c_i}, we note that $\bar{a}_{c_i}$ is the part of the acceleration $\dot{v}_{c_i}$ which is not linear in $\dot{\Omega}^j$ and $\dot{U}^j$. Note that $\bar{C}^i$ is a "Coriolis-type" or "centrifugal-type" force on Body i, and $\bar{C}^{\bar{j}}$ is this force on system $\bar{j}$. $\bar{E}^{\bar{j}}_{h_j}$ is a combined "Euler coupling" torque plus moment of Coriolis-type and centrifugal-type force. Of course, $\bar{X}$ and $\bar{Y}$ are only "fictitious forces"; the true force is $\bar{K}$.

The $2n$ equations in Eq. (27) can be written out as follows:

$$\dot{H}^{\bar{i}}_{h_i}+\tilde{v}_{h_i}\bullet P^{\bar{i}}=L^{\bar{i}}_{h_i} \tag{33}$$

$$\dot{P}^{\bar{i}}=F^{\bar{i}} \tag{34}$$

Note that these equations are essentially Eqs. (1.64) and (1.59) of Ref. 15, written for System $\bar{i}$ with respect to the moving point h_i. Similarly, for the $2n$ equations in Eq. (28), we have

$$\sum_{j=1}^{n}(\vec{\vec{I}}^{\overline{ij}}_{h_i h_j}\bullet\dot{\Omega}^j+\vec{S}_{c_{\overline{ij}}h_i}\bullet\dot{U}^j)+\bar{E}^{\bar{i}}_{h_i}=L^{\bar{i}}_{h_i} \tag{35a}$$

$$\sum_{j=1}^{n}(\vec{S}^t_{c_{\overline{ji}}h_j}\bullet\dot{\Omega}^j+\vec{M}^{\overline{ij}}\bullet\dot{U}^j)+\bar{C}^{\bar{i}}=F^{\bar{i}} \tag{35b}$$

The transformed equations of motion, $\dot{\bar{G}}+\bar{X}=\bar{K}$ and $\bar{\mu}\bullet\dot{\bar{\sigma}}+\bar{Y}=\bar{K}$, are not really useful in themselves, because they allow six degrees of freedom between all of the bodies of the tree configuration. If there are really six degrees of freedom between all of the bodies, then it is much simpler to use the primitive equations of motion $\dot{G}+X=K$ or $\mu\bullet\dot{\sigma}+Y=K$. The real utility of the transformed equations is that, in obtaining them, we introduced a number of important vector and dyadic variables which can be used as intermediate or auxiliary variables when the final equations (with less than six degrees of freedom between some of the bodies) are obtained.

V. Separation of Free and Constrained Motion

To facilitate the imposition of constraints in relative motion, we expand Ω^i, U^i, $H^{\bar{i}}_{h_i}$, and $P^{\bar{i}}$ into scalar components as follows:

$$\Omega^i\ =\Omega^i_{\gamma^i_1}\hat{\gamma}^i_1+\Omega^i_{\gamma^i_2}\hat{\gamma}^i_2+\Omega^i_{\gamma^i_3}\hat{\gamma}^i_3 \tag{36a}$$

$$U^i\ =U^i_{\delta^i_1}\hat{\delta}^i_1+U^i_{\delta^i_2}\hat{\delta}^i_2+U^i_{\delta^i_3}\hat{\delta}^i_3 \tag{36b}$$

$$H^{\bar{i}}_{h_i}=H^{\bar{i}}_{h_i,\gamma^{i*}_1}\gamma^{i*}_1+H^{\bar{i}}_{h_i,\gamma^{i*}_2}\gamma^{i*}_2+H^{\bar{i}}_{h_i,\gamma^{i*}_3}\gamma^{i*}_3 \tag{36c}$$

$$P^{\bar{i}}\ =P^{\bar{i}}_{\delta^i_1}\hat{\delta}^i_1+P^{\bar{i}}_{\delta^i_2}\hat{\delta}^i_2+P^{\bar{i}}_{\delta^i_3}\hat{\delta}^i_3 \tag{36d}$$

$L^{\bar{i}}_{h_i}$ and $F^{\bar{i}}$ are expanded similarly to $H^{\bar{i}}_{h_i}$ and $P^{\bar{i}}$, respectively. $\hat{\gamma}^i_s$ for $s=1$ to 3 are three generally nonorthogonal unit vectors at h_i in Body i, and $\hat{\delta}^i_s$ for $s=1$ to 3 are three orthogonal unit vectors at b_i in Body i. γ^{i*}_s is the reciprocal vector to $\hat{\gamma}^i_s$, and hence $\gamma^{i*}_s\bullet\hat{\gamma}^i_s=1$, and $\gamma^{i*}_r\bullet\hat{\gamma}^i_s=0$ if $r\neq s$. Now introduce the following column matrices:

$$\Gamma^i=\begin{bmatrix}\hat{\gamma}^i_1\\ \hat{\gamma}^i_2\\ \hat{\gamma}^i_3\end{bmatrix},\ \Gamma^{i*}=\begin{bmatrix}\gamma^{i*}_1\\ \gamma^{i*}_2\\ \gamma^{i*}_3\end{bmatrix},\ \Delta^i=\begin{bmatrix}\hat{\delta}^i_1\\ \hat{\delta}^i_2\\ \hat{\delta}^i_3\end{bmatrix} \tag{37}$$

$$\Omega^i_{\Gamma^i}=\begin{bmatrix}\Omega^i_{\gamma^i_1}\\ \Omega^i_{\gamma^i_2}\\ \Omega^i_{\gamma^i_3}\end{bmatrix},\quad U^i_{\Delta^i}=\begin{bmatrix}U^i_{\delta^i_1}\\ U^i_{\delta^i_2}\\ U^i_{\delta^i_3}\end{bmatrix} \tag{38a}$$

$$H^{\bar i}_{h_i,\Gamma^{i*}} = \begin{bmatrix} H^{\bar i}_{h_i,\gamma_1^{i*}} \\ H^{\bar i}_{h_i,\gamma_2^{i*}} \\ H^{\bar i}_{h_i,\gamma_3^{i*}} \end{bmatrix}, \quad P^{\bar i}_{\Delta^i} = \begin{bmatrix} P^{\bar i}_{\delta_1^i} \\ P^{\bar i}_{\delta_2^i} \\ P^{\bar i}_{\delta_3^i} \end{bmatrix} \tag{38b}$$

Equations (36) now become

$$\boldsymbol{\Omega}^i = \boldsymbol{\Gamma}^{i^t}\Omega^i_{\Gamma^i}, \qquad \boldsymbol{U}^i = \boldsymbol{\Delta}^{i^t} U^i_{\Delta^i} \tag{39a}$$

$$\boldsymbol{H}^{\bar i}_{h_i} = \boldsymbol{\Gamma}^{i*^t} H^{\bar i}_{h_i,\Gamma^{i*}}, \qquad \boldsymbol{P}^{\bar i} = \boldsymbol{\Delta}^{i^t} P^{\bar i}_{\Delta^i} \tag{39b}$$

Now making use of

$$\boldsymbol{\Gamma}^{i*}\cdot\boldsymbol{\Gamma}^{i^t} = 1_3, \quad \boldsymbol{\Delta}^i\cdot\boldsymbol{\Delta}^{i^t} = 1_3, \quad \boldsymbol{\Gamma}^i\cdot\boldsymbol{\Gamma}^{i*^t} = 1_3 \tag{40}$$

where 1_3 is the 3×3 identity matrix, we can invert Eqs. (39) to yield

$$\Omega^i_{\Gamma^i} = \boldsymbol{\Gamma}^{i*}\cdot\boldsymbol{\Omega}^i, \qquad U^i_{\Delta^i} = \boldsymbol{\Delta}^i\cdot\boldsymbol{U}^i \tag{41a}$$

$$H^{\bar i}_{h_i,\Gamma^{i*}} = \boldsymbol{\Gamma}^i\cdot\boldsymbol{H}^{\bar i}_{h_i}, \qquad P^{\bar i}_{\Delta^i} = \boldsymbol{\Delta}^i\cdot\boldsymbol{P}^{\bar i} \tag{41b}$$

These equations can be inverted again, to get back to Eqs. (39), by making use of

$$\boldsymbol{\Gamma}^{i^t}\boldsymbol{\Gamma}^{i*} = \overleftrightarrow{1}, \qquad \boldsymbol{\Delta}^{i^t}\boldsymbol{\Delta}^i = \overleftrightarrow{1}, \quad \boldsymbol{\Gamma}^{i*^t}\boldsymbol{\Gamma}^i = \overleftrightarrow{1} \tag{42}$$

where $\overleftrightarrow{1}$ is the identity dyadic.

Now suppose that the "gimbal axes" or "Euler angle axes" $\hat{\gamma}^i_s$ are arranged so that, if there is one free rotational degree of freedom for Body i, this degree of freedom is about $\hat{\gamma}^i_1$; if there are two free rotational degrees of freedom, these degrees of freedom are about $\hat{\gamma}^i_1$ and $\hat{\gamma}^i_2$. Similarly, suppose that the "displacement axes" $\hat{\delta}^i_s$ are arranged so that, if there is one free translational degree of freedom for Body i, this degree of freedom is along $\hat{\delta}^i_1$; if there are two free translational degrees of freedom, these degrees of freedom are along $\hat{\delta}^i_1$ and $\hat{\delta}^i_2$. This orderly separation of free and constrained axes allows us to write $\boldsymbol{\Gamma}^i$, $\boldsymbol{\Gamma}^{i*}$, and $\boldsymbol{\Delta}^i$ as follows:

$$\boldsymbol{\Gamma}^i = \begin{bmatrix} \boldsymbol{\Gamma}^i_f \\ \boldsymbol{\Gamma}^i_c \end{bmatrix}, \quad \boldsymbol{\Gamma}^{i*} = \begin{bmatrix} \boldsymbol{\Gamma}^{i*}_f \\ \boldsymbol{\Gamma}^{i*}_c \end{bmatrix}, \quad \boldsymbol{\Delta}^i = \begin{bmatrix} \boldsymbol{\Delta}^i_f \\ \boldsymbol{\Delta}^i_c \end{bmatrix} \tag{43}$$

where the subscript f denotes the free axes, and the subscript c denotes the constrained axes. Similarly, we can separate the components of $\boldsymbol{\Omega}^i$, $\boldsymbol{U}^i$, $\boldsymbol{H}^{\bar i}_{h_i}$, and $\boldsymbol{P}^{\bar i}$ as follows:

$$\Omega^i_{\Gamma^i} = \begin{bmatrix} \Omega^i_f \\ \Omega^i_c \end{bmatrix}, \quad U^i_{\Delta^i} = \begin{bmatrix} U^i_f \\ U^i_c \end{bmatrix}, \quad H^{\bar i}_{h_i,\Gamma^{i*}} = \begin{bmatrix} H^{\bar i}_f \\ H^{\bar i}_c \end{bmatrix}, \quad P^{\bar i}_{\Delta^i} = \begin{bmatrix} P^{\bar i}_f \\ P^{\bar i}_c \end{bmatrix} \tag{44}$$

Equations (36) now can be expressed as [compare with Eqs. (39)]

$$\boldsymbol{\Omega}^i = \boldsymbol{\Gamma}^{i^t}_f\,\Omega^i_f + \boldsymbol{\Gamma}^{i^t}_c\,\Omega^i_c \tag{45a}$$

$$\boldsymbol{U}^i = \boldsymbol{\Delta}^{i^t}_f\,U^i_f + \boldsymbol{\Delta}^{i^t}_c\,U^i_c \tag{45b}$$

$$\boldsymbol{H}^{\bar i}_{h_i} = \boldsymbol{\Gamma}^{i*^t}_f\,H^{\bar i}_f + \boldsymbol{\Gamma}^{i*^t}_c\,H^{\bar i}_c \tag{45c}$$

$$\boldsymbol{P}^{\bar i} = \boldsymbol{\Delta}^{i^t}_f\,P^{\bar i}_f + \boldsymbol{\Delta}^{i^t}_c\,P^{\bar i}_c \tag{45d}$$

Inversely [compare with Eqs. (41)],

$$\Omega^i_f = \boldsymbol{\Gamma}^{i*}_f\cdot\boldsymbol{\Omega} \qquad \Omega^i_c = \boldsymbol{\Gamma}^{i*}_c\cdot\boldsymbol{\Omega} \tag{46a}$$

$$U^i_f = \boldsymbol{\Delta}^i_f\cdot\boldsymbol{U}^i \qquad U^i_c = \boldsymbol{\Delta}^i_c\cdot\boldsymbol{U}^i \tag{46b}$$

$$H^{\bar i}_f = \boldsymbol{\Gamma}^i_f\cdot\boldsymbol{H}^{\bar i}_{h_i} \qquad H^{\bar i}_c = \boldsymbol{\Gamma}^i_c\cdot\boldsymbol{H}^{\bar i}_{h_i} \tag{46c}$$

$$P^{\bar i}_f = \boldsymbol{\Delta}^i_f\cdot\boldsymbol{P}^{\bar i} \qquad P^{\bar i}_c = \boldsymbol{\Delta}^i_c\cdot\boldsymbol{P}^{\bar i} \tag{46d}$$

$\boldsymbol{L}^{\bar i}_{h_i}$ and $\boldsymbol{F}^{\bar i}$ are expanded similarly.

A. Velocity Transformation

We are now in a position to define the free variables $\hat{\sigma}_f$, $\hat{G}_f$, $\hat{K}_f$ and the constrained variables $\hat{\sigma}_c$, $\hat{G}_c$, $\hat{K}_c$. Define $\hat{A}$ and $\hat{\sigma}$ as follows:

$$\bar{\sigma} = \hat{A}\hat{\sigma} \tag{47}$$

where $\hat{A}$ is a rectangular matrix of vectors, and $\hat{\sigma}$ is a column matrix of scalars. Now define $\hat{\sigma}_f$ and $\hat{\sigma}_c$ as follows:

$$\hat{\sigma}_f = \begin{bmatrix} \Omega^1_f \\ \vdots \\ \Omega^n_f \\ U^1_c \\ \vdots \\ U^n_f \end{bmatrix}, \quad \hat{\sigma}_c = \begin{bmatrix} \Omega^1_c \\ \vdots \\ \Omega^n_c \\ U^1_c \\ \vdots \\ U^n_c \end{bmatrix} \tag{48}$$

Then Eq. (47) can be written readily in partitioned form as follows:

$$\bar{\sigma} = [\hat{A}_f \quad \hat{A}_c]\begin{bmatrix} \hat{\sigma}_f \\ \hat{\sigma}_c \end{bmatrix} = \hat{A}_f\hat{\sigma}_f + \hat{A}_c\hat{\sigma}_c \tag{49}$$

$$\hat{\sigma} = \hat{B}\cdot\bar{\sigma} = \begin{bmatrix} \hat{B}_f \\ \hat{B}_c \end{bmatrix}\cdot\bar{\sigma} = \begin{bmatrix} \hat{\sigma}_f \\ \hat{\sigma}_c \end{bmatrix} \tag{50}$$

where $\hat{B}$ is a rectangular matrix of vectors, and $\hat{B}_f$ and $\hat{B}_c$ are block diagonal matrices obtained by comparing Eqs. (46a) and (46b) with Eq. (50).

Evidently, $\hat{A}$ is a matrix of unit vectors. $\hat{B}$ can be formed by taking the transpose of this matrix and then replacing each of the unit vectors by their reciprocal vectors: symbolically, $\hat{B} = \hat{A}^{t*}$. Note that $\hat{B}\cdot\hat{A} = 1_{6n}$, the $6n\times 6n$ identity matrix, whereas $\hat{A}\hat{B}$ is an identity with $2n$ $\overleftrightarrow{1}$'s on the diagonal.

B. Induced Transformations

Now define $\hat{G}$ and $\hat{K}$ so that the kinetic energy T and its time derivative $\dot{T}$ maintain the general form

$$T = \tfrac{1}{2}\bar{G}^t\cdot\bar{\sigma} = \tfrac{1}{2}\hat{G}^t\hat{\sigma} = \tfrac{1}{2}(\hat{G}^t_f\hat{\sigma}_f + \hat{G}^t_c\hat{\sigma}_c) \tag{51}$$

$$\dot{T} = \bar{K}^t\cdot\bar{\sigma} = \hat{K}^t\hat{\sigma} = \hat{K}^t_f\hat{\sigma}_f + \hat{K}^t_c\hat{\sigma}_c \tag{52}$$

This requires that $\hat{G}$ be defined by

$$\hat{G} = \hat{A}^t\cdot\bar{G} = \begin{bmatrix} \hat{A}^t_f \\ \hat{A}^t_c \end{bmatrix}\cdot\bar{G} = \begin{bmatrix} \hat{G}_f \\ \hat{G}_c \end{bmatrix} \tag{53}$$

Carrying out the indicated operations yields

$$\hat{G}_f = \begin{bmatrix} H^{\bar 1}_f \\ \vdots \\ H^{\bar n}_f \\ P^{\bar 1}_f \\ \vdots \\ P^{\bar n}_f \end{bmatrix}, \quad \hat{G}_c = \begin{bmatrix} H^{\bar 1}_c \\ \vdots \\ H^{\bar n}_c \\ P^{\bar 1}_c \\ \vdots \\ P^{\bar n}_c \end{bmatrix} \tag{54}$$

Similar equations are obtained for $\hat{K}$, with L's and F's replacing H's and P's. Note that $\hat{\sigma}$, $\hat{G}$, and $\hat{K}$ are column matrices of scalars.

It also is necessary to define $\hat{\mu}$ by

$$\hat{\mu}=\hat{A}^{t}\cdot\bar{\mu}\cdot\hat{A}=\begin{bmatrix}\hat{A}_f^t\\ \hat{A}_c^t\end{bmatrix}\cdot\bar{\mu}\cdot[\hat{A}_f \quad \hat{A}_c]$$

$$=\begin{bmatrix}\hat{A}_f^t\cdot\bar{\mu}\cdot\hat{A}_f & \hat{A}_f^t\cdot\bar{\mu}\cdot\hat{A}_c\\ \hat{A}_c^t\cdot\bar{\mu}\cdot\hat{A}_f & \hat{A}_c^t\cdot\bar{\mu}\cdot\hat{A}_c\end{bmatrix}=\begin{bmatrix}\hat{\mu}_{ff} & \hat{\mu}_{fc}\\ \hat{\mu}_{cf} & \hat{\mu}_{cc}\end{bmatrix} \tag{55}$$

Carrying out the indicated operation yields

$$\hat{\mu}_{rs}=\hat{A}_r^t\cdot\bar{\mu}\cdot\hat{A}_s=\begin{bmatrix}\hat{I}_{rs}^{\bar{l}\bar{l}} & \cdots & \hat{I}_{rs}^{\bar{l}\bar{n}} & \hat{S}_{rs}^{\bar{l}\bar{l}} & \cdots & \hat{S}_{rs}^{\bar{l}\bar{n}}\\ \vdots & & & & & \\ \hat{I}_{rs}^{\bar{n}\bar{l}} & \cdots & \hat{I}_{rs}^{\bar{n}\bar{n}} & \hat{S}_{rs}^{\bar{n}\bar{l}} & \cdots & \hat{S}_{rs}^{\bar{n}\bar{n}}\\ \hat{S}_{sr}^{\bar{l}\bar{l}^t} & \cdots & \hat{S}_{sr}^{\bar{n}\bar{l}^t} & \hat{M}_{rs}^{\bar{l}\bar{l}} & \cdots & \hat{M}_{rs}^{\bar{l}\bar{n}}\\ \vdots & & & & & \\ \hat{S}_{sr}^{\bar{l}\bar{n}^t} & \cdots & \hat{S}_{sr}^{\bar{n}\bar{n}^t} & \hat{M}_{rs}^{\bar{n}\bar{l}} & \cdots & \hat{M}_{rs}^{\bar{n}\bar{n}}\end{bmatrix} \tag{56}$$

where r and s are equal to f or c, and where

$$\hat{I}_{rs}^{\bar{i}\bar{j}}=\mathbf{\Gamma}_r^i\cdot\bar{I}_{h_ih_j}^{\bar{i}\bar{j}}\cdot\mathbf{\Gamma}_s^{j^t} \tag{57}$$

$$\hat{S}_{rs}^{\bar{i}\bar{j}}=\mathbf{\Gamma}_r^i\cdot\bar{S}_{c_{\bar{i}\bar{j}}h_i}\cdot\Delta_s^{j^t} \tag{58}$$

$$\hat{M}_{rs}^{\bar{i}\bar{j}}=\mathbf{\Delta}_r^i\cdot\bar{M}^{\bar{i}\bar{j}}\cdot\Delta_s^{j^t} \tag{59}$$

The relationship $\hat{G}=\hat{\mu}\hat{\sigma}$ has the inverse relationship $\hat{\sigma}=\hat{\nu}\hat{G}$, where $\hat{\nu}$ is defined by

$$\hat{\nu}=\hat{B}\cdot\bar{\nu}\cdot\hat{B}^t=\begin{bmatrix}\hat{B}_f\\ \hat{B}_c\end{bmatrix}\cdot\bar{\nu}\cdot[\hat{B}_f^t \quad \hat{B}_c^t]$$

$$=\begin{bmatrix}\hat{B}_f\cdot\bar{\nu}\cdot\hat{B}_f^t & \hat{B}_f^t\cdot\bar{\nu}\cdot\hat{B}_c^t\\ \hat{B}_c\cdot\bar{\nu}\cdot\hat{B}_f^t & \hat{B}_c\cdot\bar{\nu}\cdot\hat{B}_c^t\end{bmatrix}=\begin{bmatrix}\hat{\nu}_{ff} & \hat{\nu}_{fc}\\ \hat{\nu}_{cf} & \hat{\nu}_{cc}\end{bmatrix} \tag{60}$$

C. Separated Equation of Motion

The momentum formulation equation of motion now is given by

$$\dot{\hat{G}}+\hat{X}=\hat{K} \quad \text{where} \quad \hat{X}=\hat{A}^t\cdot\bar{X}-\dot{\hat{A}}^t\cdot\bar{G} \tag{61}$$

and the velocity formulation equation of motion is

$$\hat{\mu}\dot{\hat{\sigma}}+\hat{Y}=\hat{K} \tag{62a}$$

where

$$\hat{Y}=\hat{A}^t\cdot(\bar{Y}+\bar{\mu}\cdot\dot{\hat{A}}\hat{\sigma}) \tag{62b}$$

It actually is more convenient to determine $\hat{Y}$ from Eq. (62b) with $\bar{\mu}$ expanded. Thus,

$$\hat{Y}=\hat{A}^t\cdot\{A^t\cdot[Y+\mu(\dot{A}\cdot\bar{\sigma}+A\cdot\dot{\hat{A}}\hat{\sigma})]\} \tag{63}$$

Partitioning Eqs. (61) and (62) into free and constrained parts yields

$$\dot{\hat{G}}_r+\hat{X}_r=\hat{K}_r \quad \text{where} \quad \hat{X}_r=\hat{A}_r^t\cdot\bar{X}-\dot{\hat{A}}_r^t\cdot\bar{G} \tag{64}$$

$$\hat{\mu}_{rf}\dot{\hat{\sigma}}_f+\hat{\mu}_{rc}\dot{\hat{\sigma}}_c+\hat{Y}_r=\hat{K}_r \quad \text{where} \quad \hat{Y}_r=\hat{A}_r^t\cdot(\bar{Y}+\bar{\mu}\cdot\dot{\hat{A}}\hat{\sigma}) \tag{65}$$

where $r=f$ or c, and where $\hat{X}_r$ and $\hat{Y}_r$ are given by†

$$\hat{X}_r=\begin{bmatrix}\mathbf{\Gamma}_r^l\cdot\tilde{v}_{h_l}\cdot\mathbf{P}^{\bar{l}}-\dot{\mathbf{\Gamma}}_r^l\cdot\mathbf{H}_{h_l}^{\bar{l}}\\ \vdots\\ \mathbf{\Gamma}_r^n\cdot\tilde{v}_{h_n}\cdot\mathbf{P}^{\bar{n}}-\dot{\mathbf{\Gamma}}_r^n\cdot\mathbf{H}_{h_n}^{\bar{n}}\\ -\dot{\mathbf{\Delta}}_r^l\cdot\mathbf{P}^{\bar{l}}\\ \vdots\\ -\dot{\mathbf{\Delta}}_r^n\cdot\mathbf{P}^{\bar{n}}\end{bmatrix}, \hat{Y}_r=\begin{bmatrix}\mathbf{\Gamma}_r^l\cdot\bar{\bar{E}}_{h_l}^{\bar{l}}\\ \vdots\\ \mathbf{\Gamma}_r^n\cdot\bar{\bar{E}}_{h_n}^{\bar{n}}\\ \mathbf{\Delta}_r^l\cdot\bar{\bar{C}}^{\bar{l}}\\ \vdots\\ \mathbf{\Delta}_r^n\cdot\bar{\bar{C}}^{\bar{n}}\end{bmatrix} \tag{66}$$

where

$$\bar{\bar{E}}_{h_j}^{\bar{j}}=\sum_{i\in\bar{j}}(\bar{\bar{E}}_{c_i}^i+\tilde{R}_{c_ih_j}\cdot\bar{\bar{C}}^i) \tag{67}$$

$$\bar{\bar{E}}_{c_i}^i=E_{c_i}^i+\vec{I}_{c_i}^i\cdot\bar{\bar{\alpha}}^i \tag{68}$$

$$\bar{\bar{\alpha}}^i=\sum_{k\in\hat{i}}\dot{\mathbf{\Gamma}}^{k^t}\Omega_{\Gamma}^k{}_k \tag{69}$$

$$\bar{\bar{C}}^i=M^i\sum_{k\in\hat{i}}(\dot{\tilde{R}}_{c_ih_k}^t\cdot\mathbf{\Omega}^k+\tilde{R}_{c_ih_k}^t\cdot\dot{\mathbf{\Gamma}}^{k^t}\Omega_{\Gamma k}^k+\dot{\mathbf{\Delta}}^{k^t}U_{\Delta k}^k) \tag{70}$$

$$=M^i\bar{\bar{a}}_{c_i}$$

$$\bar{\bar{C}}^{\bar{j}}=\sum_{i\in\bar{j}}\bar{\bar{C}}^i \tag{71}$$

Note that M^i times the sum over k for the first term in Eq. (70) yields $\bar{C}^i$ of Eq. (31). Also note that $\bar{\bar{a}}_{c_i}$ is the part of the translational acceleration $\dot{v}_{c_i}$ which is not linear in $\dot{\Omega}_{\Gamma}^k{}_k$ and $\dot{U}_{\Delta}^k{}_k$. Similarly, $\bar{\bar{\alpha}}_i$ is the part of rotational acceleration $\dot{\omega}^i$ which is not linear in $\dot{\Omega}_{\Gamma}^k{}_k$ and $\dot{U}_{\Delta}^k{}_k$. It is clear that $\hat{X}_r$ is conceptually simpler than $\hat{Y}_r$.

Equations (64) and (65) each represent two equations: one for the free variables (for $r=f$), and one for the constrained variables. The constrained variables can be determined algebraically from the free variables, and therefore in a dynamics simulation we need only to "integrate" the free variables equation. In the momentum formulation, the required differential equation is

$$\dot{\hat{G}}_f+\hat{X}_f=\hat{K}_f \tag{72}$$

together with the algebraic equation

$$\hat{G}_f=\hat{\mu}_{ff}\hat{\sigma}_f+\hat{\mu}_{fc}\hat{\sigma}_c \tag{73}$$

In effect, it is necessary to invert $\hat{\mu}_{ff}$ in order to obtain $\hat{\sigma}_f$ in terms of the "known" quantities $\hat{G}_f$ (known from "integration") and $\hat{\sigma}_c$ (prescribed). In the velocity formulation, the required differential equation is

$$\hat{\mu}_{ff}\dot{\hat{\sigma}}_f+\hat{\mu}_{fc}\dot{\hat{\sigma}}_c+\hat{Y}_f=\hat{K}_f \tag{74}$$

Note that in this formulation we also must effectively invert $\hat{\mu}_{ff}$; in addition, we must generate $\dot{\hat{\sigma}}_c$.

The constraint force $\hat{K}_c$ is not required in either formulation. Should it be desired to generate $\hat{K}_c$, it can be done in the momentum formulation from

$$\hat{K}_c=\dot{\hat{G}}_c+\hat{X}_c \tag{75}$$

†Note that $\dot{\Delta}_r^l=0$; it is retained in $\hat{X}_r$ in Eq. (66) merely to show the general pattern.

and in the velocity formulation from

$$\hat{K}_c = \hat{\mu}_{cf}\dot{\hat{\sigma}}_f + \hat{\mu}_{cc}\dot{\hat{\sigma}}_c + \hat{Y}_c \tag{76}$$

Note that, in a velocity formulation simulation, all of the quantities required to generate the constraint force $\hat{K}_c$ are already "available," whereas in a momentum formulation it is necessary to generate $\dot{\hat{G}}_c$. We shall give an alternative expression for $\hat{K}_c$ in the momentum formulation in the next section [see Eq. (100)].

The momentum formulation equations presented herein differ from the equations of Russell[16,17] which involve velocities and momenta relative the composite center of mass. Such equations can be obtained by transforming from the preceding transformed velocity $\bar{\sigma}$ to $\bar{\sigma}'$, which is the same as $\bar{\sigma}$ except that U^I is replaced by v_{c_T}, the velocity of the composite center of mass. $\hat{\sigma}$ then is replaced by a similar $\hat{\sigma}'$; similarly, $\hat{G}$ and $\hat{K}$ are replaced by $\hat{G}'$ and $\hat{K}'$. In fact, Russell refers to $\bar{G}'$ and $\hat{G}'$ as "primed momentum." In such a formulation, one also obtains a $\hat{\mu}'$, $\hat{\nu}'$, etc., but, for computational efficiency, Russell avoids analytic generation of $\hat{\mu}'$.

VI. Coupling of Free and Constrained Motion

As an alternative to transforming to free and constrained variables, we can leave the equations of motion and equations of constraints in a coupled form and then solve the equations of motion and equations of constraints simultaneously. Recall that we started out with $\dot{G}+X=K$ or $\mu\bullet\dot{\sigma}+Y=K$ and then made the transformation

$$\sigma = \hat{\mathfrak{A}}\hat{\sigma} \quad \text{where} \quad \hat{\mathfrak{A}} = A\bullet\hat{A} \tag{77}$$

This can be written as

$$\sigma = [\hat{\mathfrak{A}}_f \;\; \hat{\mathfrak{A}}_c]\begin{bmatrix}\hat{\sigma}_f\\ \hat{\sigma}_c\end{bmatrix} = \hat{\mathfrak{A}}_f\hat{\sigma}_f + \hat{\mathfrak{A}}_c\hat{\sigma}_c \tag{78}$$

where $\hat{\mathfrak{A}}_s = A\bullet\hat{A}_s$ for $s=f$ and c. Inversely, we have

$$\hat{\sigma} = \hat{\mathfrak{B}}\bullet\sigma = \begin{bmatrix}\hat{\mathfrak{B}}_f\\ \hat{\mathfrak{B}}_c\end{bmatrix}\bullet\sigma = \begin{bmatrix}\hat{\sigma}_f\\ \hat{\sigma}_c\end{bmatrix} \tag{79}$$

where $\hat{\mathfrak{B}} = \hat{B}\bullet B$ and $\hat{\mathfrak{B}}_s = \hat{B}_s\bullet B$ for $s=f$ and c. Since we consider $\hat{\sigma}_c$ to be prescribed, we effectively have the equation of constraint

$$\hat{\mathfrak{B}}_c\bullet\sigma = \hat{\sigma}_c \tag{80}$$

There are n_c scalar elements in $\hat{\sigma}_c$, and thus there are n_c scalar equations of constraints.

From the equation $\sigma = \hat{\mathfrak{A}}\hat{\sigma}$, we get the equation

$$\hat{K} = \hat{\mathfrak{A}}^t\bullet K = \begin{bmatrix}\hat{\mathfrak{A}}_f^t\\ \hat{\mathfrak{A}}_c^t\end{bmatrix}\bullet K = \begin{bmatrix}\hat{K}_f\\ \hat{K}_c\end{bmatrix} \tag{81}$$

Inversely, we have

$$K = \hat{\mathfrak{B}}^t\hat{K} = [\hat{\mathfrak{B}}_f^t \;\; \hat{\mathfrak{B}}_c^t]\begin{bmatrix}\hat{K}_f\\ \hat{K}_c\end{bmatrix} = \hat{\mathfrak{B}}_f^t\hat{K}_f + \hat{\mathfrak{B}}_c^t\hat{K}_c \tag{82}$$

Thus, we can write

$$K = K^a + K^c \tag{83}$$

where

$$K^a = \hat{\mathfrak{B}}_f^t\hat{K}_f, \qquad K^c = \hat{\mathfrak{B}}_c^t\hat{K}_c \tag{84}$$

K^a is the "applied" part of K, and K^c is the "constraint" part of K.

If the multibody configuration is not a tree, it is not a simple matter to find the transformation operator $\hat{\mathfrak{A}}$ and the appropriate velocity $\hat{\sigma}$ with free elements $\hat{\sigma}_f$ and constrained elements $\hat{\sigma}_c$. However, it usually is simple to obtain an expression of the form

$$\tilde{\mathfrak{B}}_c\bullet\sigma = \tilde{\sigma}_c \tag{85}$$

for the constraints of the multibody configuration. Here $\tilde{\mathfrak{B}}_c$ is some (primitive) velocity constraint transformation operator, and $\tilde{\sigma}_c$ is some prescribed constraint velocity. In a tree configuration, we simply take $\tilde{\mathfrak{B}}_c$ to be $\hat{\mathfrak{B}}_c$, and we take $\tilde{\sigma}_c$ to be $\hat{\sigma}_c$. From the general properties of Lagrange multipliers, it now follows that we can write

$$K = K^a + K^c \quad \text{where} \quad K^c = \tilde{\mathfrak{B}}_c^t\tilde{K}_c \tag{86}$$

where $\tilde{K}_c$ is the Lagrange multiplier column matrix (usually denoted by λ). In the case of a tree configuration, we take $\tilde{K}_c$ to be $\hat{K}_c$. Thus, the momentum formulation equations take the form

$$\dot{G} + X = K^a + \tilde{\mathfrak{B}}_c^t\tilde{K}_c \tag{87}$$

and the velocity formulation equations take the form

$$\mu\bullet\dot{\sigma} + Y = K^a + \tilde{\mathfrak{B}}_c^t\tilde{K}_c \tag{88}$$

In either formulation, we also need the constraint relationship in Eq. (85). We now shall examine what is required to determine $\tilde{K}_c$ in either formulation.

A. Velocity Formulation

In the velocity formulation, the simultaneous equations that must be solved are

$$\begin{bmatrix}\mu & \tilde{\mathfrak{B}}_c^t\\ \tilde{\mathfrak{B}}_c & O\end{bmatrix}\bullet\begin{bmatrix}\dot{\sigma}\\ -\tilde{K}_c\end{bmatrix} = \begin{bmatrix}K^a - Y\\ \dot{\tilde{\sigma}}_c - \dot{\tilde{\mathfrak{B}}}_c\bullet\sigma\end{bmatrix} \tag{89}$$

where the second row of this matrix equation was obtained by taking the time derivative of Eq. (85). Inverting this relationship yields

$$\begin{bmatrix}\dot{\sigma}\\ -\tilde{K}_c\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\bullet\begin{bmatrix}K^a - Y\\ \dot{\tilde{\sigma}}_c - \dot{\tilde{\mathfrak{B}}}_c\bullet\sigma\end{bmatrix} \tag{90}$$

where

$$d = -(\tilde{\mathfrak{B}}_c\bullet\nu\bullet\tilde{\mathfrak{B}}_c^t)^{-1}, \qquad a = \nu + \nu\bullet\tilde{\mathfrak{B}}_c^t d\tilde{\mathfrak{B}}_c\bullet\nu \tag{91a}$$

$$b = -\nu\bullet\tilde{\mathfrak{B}}_c^t d, \qquad c = -d\tilde{\mathfrak{B}}_c\bullet\nu \tag{91b}$$

Hence,

$$\tilde{K}_c = -c\bullet(K^a - Y) - d(\dot{\tilde{\sigma}}_c - \dot{\tilde{\mathfrak{B}}}_c\bullet\sigma) \tag{92}$$

The differential equation of motion is now

$$\dot{\sigma} = a\bullet(K^a - Y) + b(\dot{\tilde{\sigma}}_c - \dot{\tilde{\mathfrak{B}}}_c\bullet\sigma) \tag{93}$$

Note that, for a tree configuration, we may take $\tilde{\mathfrak{B}}_c$ to be $\hat{\mathfrak{B}}_c$, and then d is equal to $-\hat{\nu}_{cc}^{-1}$. The evaluation of $\hat{\nu}_{cc}^{-1}$ can of course, be replaced by an evaluation of $\hat{\mu}_{ff}^{-1}$ according to the relationship

$$\hat{\nu}_{cc}^{-1} = \hat{\mu}_{cc} - \hat{\mu}_{cf}\hat{\mu}_{ff}^{-1}\hat{\mu}_{fc} \tag{94}$$

B. Momentum Formulation

In the momentum formulation, the simultaneous equations that must be solved are

$$\begin{bmatrix} 1 & \tilde{\mathcal{B}}_c^t \\ \tilde{\mathcal{B}}_c \cdot \nu & O \end{bmatrix} \cdot \begin{bmatrix} \dot{G} \\ -\tilde{K}_c \end{bmatrix} = \begin{bmatrix} K^a - X \\ \dot{\tilde{\sigma}}_c - \dot{\tilde{\mathcal{B}}}_c \cdot \sigma - \tilde{\mathcal{B}}_c \cdot \dot{\nu} \cdot G \end{bmatrix} \tag{95}$$

where the second row of this matrix equation was obtained by setting $\sigma = \nu \cdot G$ in Eq. (85) and then taking the time derivative. Inverting this relationship yields

$$\begin{bmatrix} \dot{G} \\ -\tilde{K}_c \end{bmatrix} = \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{bmatrix} \cdot \begin{bmatrix} K^a - X \\ \dot{\tilde{\sigma}}_c - \dot{\tilde{\mathcal{B}}}_c \cdot \sigma - \tilde{\mathcal{B}}_c \cdot \dot{\nu} \cdot G \end{bmatrix} \tag{96}$$

where

$$\bar{d} = -(\tilde{\mathcal{B}}_c \cdot \nu \cdot \tilde{\mathcal{B}}_c^t)^{-1} = d, \quad \bar{a} = 1 + \tilde{\mathcal{B}}_c^t \bar{d} \tilde{\mathcal{B}}_c \cdot \nu = \mu \cdot a \tag{97a}$$

$$\bar{b} = -\tilde{\mathcal{B}}_c^t \bar{d} = \mu \cdot b, \quad \bar{c} = -\bar{d}\tilde{\mathcal{B}}_c \cdot \nu = c \tag{97b}$$

Hence,

$$\tilde{K}_c = -\bar{c} \cdot (K^a - X) - \bar{d}(\dot{\tilde{\sigma}}_c - \dot{\tilde{\mathcal{B}}}_c \cdot \sigma - \tilde{\mathcal{B}}_c \cdot \dot{\nu} \cdot G) \tag{98}$$

The equation for $\dot{G}$ in Eq. (96) is actually less convenient than, though equivalent to, the original equation $\dot{G} = K^a - X + \tilde{\mathcal{B}}_c^t \tilde{K}_c$. The equivalence of $\tilde{K}_c$ in Eqs. (92) and (98) follows from

$$Y = X + \dot{\mu} \cdot \sigma = X - \mu \cdot \dot{\nu} \cdot G \tag{99}$$

Thus, whether a velocity or a momentum formulation is used, essentially the same equations are involved in determining the constraint force (or Lagrange multiplier) $\tilde{K}_c$.

Recall that, when we discussed solving for $\hat{K}_c$ in the previous section, we found that Eq. (75) for the momentum formulation was not really convenient because $\dot{\hat{G}}_c$ is not available. We now see that we can use

$$\begin{aligned} \hat{K}_c &= -\hat{\nu}_{cc}^{-1}\hat{\mathcal{B}}_c \cdot \nu \cdot (\hat{\mathcal{B}}^t_f \hat{K}_f - Y) + \hat{\nu}_{cc}^{-1}(\dot{\hat{\sigma}}_c - \dot{\hat{\mathcal{B}}}_c \cdot \sigma) \\ &= -\hat{\nu}_{cc}^{-1}\hat{\mathcal{B}}_c \cdot \nu \cdot (\hat{\mathcal{B}}^t_f \hat{K}_f - X) + \hat{\nu}_{cc}^{-1}(\dot{\hat{\sigma}}_c - \dot{\hat{\mathcal{B}}}_c \cdot \sigma - \hat{\mathcal{B}}_c \cdot \dot{\nu} \cdot G) \end{aligned} \tag{100}$$

Note that either form of Eq. (100) could be used with either the velocity or the momentum formulation.

VII. Comparison with the Literature

Table 1 lists some of the better-known references on multibody spacecraft dynamics and briefly comments on them in the light of the preceding discussion. The equations of Hooker and Margulies[18] and of Roberson and Wittenburg[6] can be obtained in two stages, as follows. First, σ is expressed in terms of $\underset{\sim}{\sigma}$, which consists of the inertial angular velocities of each body plus the inertial translational velocity of the composite center of mass; the equation of motion then is $\underset{\sim}{\mu} \cdot \dot{\underset{\sim}{\sigma}} + \underset{\sim}{Y} = \underset{\sim}{K}$, where $\sigma = \underset{\sim}{A}_1 \cdot \underset{\sim}{\sigma}$, $\underset{\sim}{\mu} = \underset{\sim}{A}_1^t \cdot \mu \cdot \underset{\sim}{A}_1$, and $\underset{\sim}{K} = \underset{\sim}{A}_1^t \cdot K$; the elements of $\underset{\sim}{\mu}$ can be expressed in terms of "barycenters" and "augmented bodies." Second, the transformed equation of motion is coupled with the relative rotation constraint equation, $\underset{\sim}{B}_2 \cdot \sigma = 0$, and the constraint torques are obtained via Lagrange multipliers. The computational aspects of the Hooker-Margulies and Roberson-Wittenburg formalisms are treated by Fleischer[19] and Farrell et al.,[20] respectively.

To relate the reference matrix ρ to the "incidence" matrix S of Roberson and Wittenburg, let ρ_1 be the first row of ρ, and let ρ_2 be the remaining $n-1$ rows. Then $\rho_2 = S^t$, and each column of S (or row of ρ_2) contains all zero elements except for one $+1$ and one -1. Similarly, the path matrix π can be related to the "ordering" matrix T of Roberson and Wittenburg (sometimes denoted by S^*). Let π_1 be the first column of π, and let π_2 be the remaining $n-1$ columns. Then $\pi_2 = T^t$, and the first column of T (or row of π_2) contains all zero elements. Since ρ and π are inverses of each other ($\rho\pi = 1_n$, the $n \times n$ identity matrix), it follows that $\rho_2\pi_2 = 1_{n-1}$; i.e., ρ_2 is a left inverse of π_2; consequently, $S^t T^t = 1_{n-1}$, or $TS = 1_{n-1}$.

The approach of Velman[21] is similar to that of Hooker-Margulies and Roberson-Wittenburg, except that inertial angular velocities are replaced by relative angular velocities (and relative linear velocities in case of point masses). The equation of motion, before imposition of the relative rotational constraint, has the form $\hat{\mu} \cdot \dot{\hat{\sigma}} + \hat{Y} = \hat{K}$, where $\hat{\sigma}$, $\hat{Y}$, and $\hat{K}$ are column matrices of scalars, and $\hat{\mu}$ is a (positive definite symmetric) matrix of scalars. Velman then eliminates the constraints via projection matrices as described by Likins[1] and by Fleischer.[19]

The equations of Palmer[22] were the first general set of equations where the constraint torques are decoupled by transformation rather than solved via Lagrange multipliers. However, Palmer's equations are restricted to configurations that are "clusters" in the sense that all of the bodies, other than Body 1, are attached to Body 1. It is interesting to note that Palmer discusses velocity, acceleration, and torque transformations, but he does not make extensive use of these transformations.

Russell[17] was the first to develop a set of transformed equations for an arbitrary tree configuration. Russell chose a momentum formulation where the dynamics state variables are the free components of the transformed momentum; however, the constrained components of momentum and the primitive and transformed velocities are retained as "intermediate" or "auxiliary" variables, so that the final equations have a particularly simple form.

Farrenkopf[23] introduced an "inductive" method of "digitally synthesizing" the dynamics equations via a "combining algorithm." The original formulation was restricted to tree configurations, but it was extended to terminal flexible bodies by Ness.[24] Their equations are essentially the "transformed" equations of this paper if left-multiplied by a nonsingular matrix; this left-multiplication is necessary to make the Farrenkopf "mass matrix" symmetric, as it is in this paper. Thus, if Farrenkopf's equation of motion is $\hat{m}\dot{\hat{\sigma}} + \hat{y} = \hat{k}$, then left-multiplying by $\hat{\alpha}^t$ yields $\hat{\mu}\dot{\hat{\sigma}} + \hat{Y} = \hat{K}$, where $\hat{\mu} = \hat{\alpha}^t\hat{m}$, $\hat{Y} = \hat{\alpha}^t\hat{y}$, and $\hat{K} = \hat{\alpha}^t\hat{k}$. If $\hat{\beta}$ is the inverse of $\hat{\alpha}$, then $\hat{k} = \hat{\beta}^t\hat{K}$; thus, $\hat{\beta}^t$ can be identified as the matrix that transforms the transformed force $\hat{K}$ of this paper to the force $\hat{k}$ of Farrenkopf; the matrix $\hat{\alpha}^t$ that symmetrizes $\hat{m}$ by left-multiplication is then the inverse of this $\hat{\beta}^t$.

The Hooker-Margulies formalism was converted to an approach that uses relative gimbal angle rates by Hooker.[25] The resulting equation of motion is $\hat{\mu}'\dot{\hat{\sigma}}' + \hat{Y}' = \hat{K}'$, where $\hat{K}'$ is the "primed force" introduced earlier in connection with Russell's "primed momentum" approach. It is interesting to note that, in the Hooker[25] formalism, Russell's primed momentum is simply $\hat{G}' = \hat{\mu}'\hat{\sigma}'$, the kinetic energy is $\hat{T} = \hat{G}'^t\sigma' = \frac{1}{2}\hat{\sigma}'^t\hat{\mu}'\hat{\sigma}'$, and the time derivative of kinetic energy is $\dot{T} = \hat{K}'^t\hat{\sigma}'$. Likins[26] extended the Hooker equations by allowing the terminal bodies to be flexible and by allowing each rigid body to contain axisymmetric rotors.

The Roberson-Wittenburg formalism was converted to an approach that uses relative gimbal angle rates by Roberson[27] and by Wittenburg.[28] Both of these extensions allow relative translation between bodies; the extension by Roberson also allows the bodies to be deformable, but the equations of motion for the deformation coordinates are not presented.

Table 1 Comparison of 10 n-body dynamics formulations

Ref.	n-body configuration	Dynamics state variables	Comments
6,18	Tree of rigid bodies; no relative translation.	Inertial angular velocities of each body; inertial linear velocity of composite.	Uses barycenters and augmented bodies; constraint torques obtained via Lagrange multipliers.
21	Tree of rigid bodies; no relative translation (except for point masses).	Relative angular velocities between bodies; inertial linear velocity of composite.	Use of relative angular velocities results in formation of composites; constraint torques removed by use of symmetric projectors.
22	Cluster of rigid bodies; no relative translation.	Relative gimbal angle rates between bodies; inertial linear velocity of composite.	Constraint torques do not appear.
17	Tree of rigid bodies; no relative translation.	Free components of angular momentum of outward composites.	Constraint torques do not appear; mass matrix is not computed explicitly.
23,24	Tree of rigid bodies[23]; terminal bodies may be flexible;[24] hinge points may be time-dependent; no relative translation.	Relative gimbal angle rates between bodies; inertial linear velocity of a material point of Body 1.	Equations are obtained inductively; constraint torques do not appear; uses a nonsymmetric "mass matrix."
25,26	Tree of rigid bodies[25]; terminal bodies may be flexible[26] no relative translation.	Relative gimbal angle rates between bodies; inertial linear velocity of composite.	Uses barycenters and augmented bodies; constraint torques do not appear.
27-30	Tree of flexible bodies; closed loops treated via Lagrange multipliers[30]; relative translation allowed.	Relative gimbal angle rates; relative displacement rates; inertial linear velocity of composite.	Uses barycenters and augmented bodies; constraint torques do not appear except with closed loops.
31-35	Tree of rigid bodies; terminal bodies may be flexible; no relative translation.	Relative gimbal angle rates; inertial linear velocity of a material point of Body 1.	Does not use barycenters and augmented bodies; constraint torques do not appear.
36	Chain of flexible bodies; no relative translation.	Relative gimbal angle rates; inertial linear velocity of a material point of Body 1.	Uses quasistatic modes plus vibration modes; constraint forces and torques do not appear.
12	Arbitrary configuration of flexible bodies; closed loops allowed; relative translation allowed; prescribed motion allowed.	Inertial angular velocities of each body; inertial linear velocity of material point of each body.	Free body equations are written for each body; all constraint forces and torques are obtained via Lagrange multipliers.

Boland et al.[29,30] also have obtained Roberson-Wittenburg-type equations in relative gimbal angle rates and relative translational rates and have described the use of this formalism in configurations with closed loops (by cutting as many loops as required to form a tree and then introducing constraints via Lagrange multipliers).

The Hooker equations have been converted to a set of equations not using "barycenters" and "augmented bodies" by Frisch,[31,32] Ho,[33,34] and Hooker.[35] In this latest version, the dynamics state variables are the relative gimbal angle rates between bodies plus the inertial linear velocity of a material point of Body 1. The resulting equation is the equation $\hat{\mu}\dot{\sigma}+\hat{Y}=\hat{K}$ of this paper (except that these papers by Frisch, Ho, and Hooker do not allow relative translation between bodies; however, the terminal bodies are allowed to be flexible; Frisch[32] treats all bodies as flexible.)

The approach of Ho[33,34] and Hooker[35] has been extended to a chain of flexible bodies by Ho et al.[36] The most interesting feature in this extension is the use of quasistatic modes plus vibration modes to describe the deformation of the flexible bodies; the use of these modes allows decoupling of the constraint forces and torques.

The formulation of Bodley et al.[12] is the most general of those in Table 1. It allows all bodies to be flexible, it allows up to six degrees of freedom between bodies (and any of these degrees of freedom may be prescribed functions of time), and it allows closed loops. The dynamics equations are retained in "primitive" or "free body" form, and the constraint forces and torques are obtained via Lagrange multipliers. It is interesting to note that Bodley et al. make fairly explicit use of velocity transformations.

Of all of the authors in Table 1, only Russell uses a momentum formulation. The transformation operator formalism was developed initially in terms of a momentum formulation,[5] and the extension to a velocity formulation was made in order to provide an overview of the alternatives in Table 1. As a matter of record, it can be noted that the use of momentum formulation also is advocated by Bodley and Park[10] and by Williams.[37]

VIII. Summary

An overview of the structure of several multibody dynamics formulations has been presented in the language of the transformation operator formalism. The following alternatives have been discussed: 1) momentum or velocity formulation; 2) separating the equations of motion from the equations of constraints, or coupling these equations; and 3) inverting $\hat{\mu}_{ff}$ or $\hat{\nu}_{cc}$ (both are positive definite symmetric).

The same type of linear simultaneous equations must be solved in the momentum formulation and in the velocity formulation. The same mass matrix and the same force appear in either formulation. In fact, the same mass matrix and force are obtained in the "matrix method" of structural analysis if the transformation matrix is updated continuously to reflect the instantaneous values of the coordinates. However, the matrix method of structural analysis does not generate the extra term $\hat{X}$ or $\hat{Y}$ because the time derivative of the transformation matrix is neglected.

The momentum and velocity formulations have been given more or less equal footing in this paper. Which of the two approaches should be used depends partially on the problem and partially on the analyst. The author has found in his own work that the momentum formulation is more convenient because usually $\hat{X}$ is simpler than $\hat{Y}$ [see Eqs. (66-71)]; however, if the acceleration $\dot{\hat{\sigma}}_f$ and the constraint force $\hat{K}_c$ are desired, the velocity formulation is probably more convenient.

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